Exam Advanced Mechanics, 9:00 – 12:00, Thursday, April 11, 2019

Olaf Scholten, KVI-CART

4 problems (total of 50 points).
The solution of every problem on a separate piece of paper with name and student number.
Some useful formulas are listed at the end.

Problem 1 (11 pnts in total)

Consider the eom of a damped oscillator which is driven by a time-dependent force, $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$.

2 pnts a. Give the complete expression for the Greens function $G(t, t')$ for the oscillator of this problem. Hint: look at the formulas at the bottom of this page.

3 pnts b. Show that the Greens function solves the eom for $F(t) = 0$ at all times except $t = t'$.

4 pnts c. Give the expressions for $x(t)$ in terms of a definite integral over exponents, sin, and/or cos functions for the case that

$$F(t) = \begin{cases} 0 & t < 0 \\ (a/\tau)\sin \pi t/\tau & 0 < t < \tau \\ 0 & t > \tau \end{cases} .$$

Pay attention to the upper and lower limits of the integral.

2 pnts d. What do you expect for $x(t)$ in the limit $\tau \to 0$ and give a short explanation.

Problem 2 (9 pnts in total)

Answer: Chapter 11.10; Coin in air

A coin is thrown up in free space, rotating with an angular frequency $\vec{\omega}$ around an arbitrary axis. The momenta of inertia along its principal axes are $I_1 = I_2$ and I_3.

2 pnts a. Which one is larger, I_1 or I_3? Give a short motivation.
Answer: $I_3 > I_2$

1 pnts b. Give the expression for the components of the angular momentum vector, $\vec{L}$, in the body-fixed frame.
Answer: $\vec{L} = (I_1\omega_1, I_1\omega_2, I_3\omega_3)$

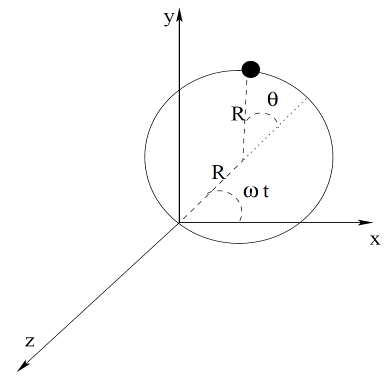
2 pnts c. The equations of motion are given by $d\vec{L}/dt = \vec{N} = 0$. Express this in terms of the momenta of inertia and the components of $\vec{\omega}$ along the principal axes and their time derivatives.
Answer: $d\vec{L}/dt_{\text{inert}} = d\vec{L}/dt_B + \vec{\omega} \times \vec{L}_B$
$(I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 = 0$; $(I_3 - I_1)\omega_1\omega_3 - I_1\dot{\omega}_2 = 0$; $I_3\dot{\omega}_3 = 0$;

2 pnts d. Show that $\omega_1(t) = A\cos\beta t$, $\omega_2(t) = A\sin\beta t$, and $\omega_3(t) = B$, is a solution of the equations of motion and give the expression for β.
Answer: $\beta = B(I_3 - I_1)/I_1$

1 pnts e. Which vector remains fixed in the inertial system.
Answer: $\vec{L} = (I_1\omega_1, I_1\omega_2, I_3\omega_3)$

Problem 3 (15 pnts in total)

A bead slides without friction on a hoop that rotates with constant angular velocity ω about an axis perpendicular to the plane of the hoop and passing through the edge of the hoop (as shown in the figure, rotating around the z-axis while the hoop is in the x-y plane). The angle θ, measures the displacement of the bead. Note that this problem ignores both friction and gravity.



4 pnts a. Give the expression for the kinetic energy of the bead.

Answer: $x = R\cos\omega t + R\cos(\omega t + \theta)$; $y = R\sin\omega t + R\sin(\omega t + \theta)$

$\dot{x} = -R\omega \sin\omega t - R(\omega + \dot{\theta})\sin(\omega t + \theta)$

$\dot{y} = R\omega\cos\omega t + R(\omega + \dot{\theta})\cos(\omega t + \theta)$

$T * 2/m = R^2\omega^2 + R^2(\omega + \dot{\theta})^2 + 2R^2\omega(\omega + \dot{\theta})\Big[\sin\omega t \sin(\omega t + \theta) + \cos\omega t\cos(\omega t + \theta)\Big]$ which reduces to $T = m\,R^2\omega^2/2 + m\,R^2(\omega + \dot{\theta})^2/2 + m\,R^2\omega(\omega + \dot{\theta})\cos\theta$

Grading: rekenfoutje: -1; geen $\theta + \omega t$: -2

1 pnts b. Show that the Lagrangian can be written in terms of the variables in the figure as

$$L = m\,R^2\omega^2/2 + m\,R^2(\omega + \dot{\theta})^2/2 + m\,R^2\omega(\omega + \dot{\theta})\cos\theta$$

2 pnts c. Give the equation of motion of the bead.

Answer: $\ddot{\theta} + \omega^2\sin\theta = 0$

2 pnts d.Show that $\theta(t) = 0$ is a solution of the EOM and determine the frequency of small oscillations around this solution.

Answer: $\ddot{\theta} + \omega^2\theta = 0$

2 pnts e. Same for the solution $\theta(t) = \pi$.

Answer: $\ddot{\theta} - \omega^2\theta = 0$

Grading: sign: -1 pnts

2 pnts f. Give the expression for the generalized momentum, p_θ.

Answer: $\partial L/d\dot{\theta} = m\,R^2(\omega + \dot{\theta}) + m\,R^2\omega\cos\theta$

2 pnts g. Determine the Hamiltonian, $H(\theta, p_\theta, t)$.

Answer: $H = P^2/(mR^2) - \omega P(1 + \cos\theta) - mR^2\omega^2/2 - [(P^2/(2mR^2) - P\omega\cos\theta + \omega^2\cos^2\theta mR^2/2] - [\omega P\cos\theta - mR^2\omega^2\cos^2\theta]$

$H = P^2/(2mR^2) - \omega P[1 + \cos\theta] + mR^2\omega^2[\cos^2\theta - 1]/2$

Grading: H=T+V: -1 pnts; geen P: -1

Problem 4 (15 pnts in total)

The action for the interaction of a particle with mass m and charge e with an electromagnetic field is given as

$$S = -\int \left[mc^2 + A^\mu u_\mu e/c\right](1/\gamma)dt \ ,$$

where $\gamma = 1/\sqrt{1-\vec{v}^2/c^2}$, $u^\mu = \frac{\partial r^\mu}{\partial \tau} = \gamma[c, \vec{v}]$, and $A^\nu(x) = [\phi(x), \vec{A}(x)]$ is the four-vector potential of the electromagnetic field.

2 pnts a. Evaluate $\partial_\mu kx$ where $kx = k^\mu x_\mu$.

1 pnts b. Evaluate $\partial^\mu kx$.

1 pnts c. Evaluate $\partial_\mu e^{ikx}$.

2 pnts d. Show that $u^2 = u^\mu u_\mu = c^2$.

1 pnts e. Evaluate $\partial^\mu u^2$.

2 pnts f. Show that the Lagrangian can be written as

$$L(\vec{x}, \vec{v}; t) = -mc^2\sqrt{1-\vec{v}^2/c^2} - e\phi(x) + e\vec{A}(x)\cdot\vec{v}/c$$

.

2 pnts g. Calculate the generalized momentum, $\vec{P}$, of the particle.

4 pnts h. Show that the equation of motion can be written as

$$\frac{d}{dt}\gamma m\vec{v} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$$

Answer:

$$\frac{d}{dt}(\gamma m\vec{v} + \frac{e}{c}\vec{A}) + e\frac{\partial\phi}{\partial\vec{x}} - \frac{e}{c}\frac{\partial\vec{A}\cdot\vec{v}}{\partial\vec{x}} = 0$$

note $\frac{d}{dt}\vec{A} = \frac{\partial\vec{A}}{\partial t} + \frac{\partial\vec{A}}{\partial x^k}\frac{d}{dt}x^k$; $\quad \vec{v}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}$ gives

$\frac{d}{dt}(\gamma m\vec{v}) = \left[-\frac{e}{c}\frac{\partial\vec{A}}{\partial t} - e\frac{\partial\phi}{\partial\vec{x}}\right] + \left[-\frac{e}{c}\frac{\partial\vec{A}}{\partial x^k}\frac{d}{dt}x^k + \frac{e}{c}\vec{v}\times\vec{B} + \frac{e}{c}(\vec{v}\cdot\vec{\nabla})\vec{A}\right]$

Possibly useful formulas:

The response of a damped oscillator $\ddot{x} + 2\beta\dot{x} + \omega_r^2 x = F(t)/m$ to a delta force at $t=0$ is $x(t) = \frac{1}{\omega_1 m}e^{-\beta t}\sin\omega_1 t$ for $t>0$ where $\omega_1 = \sqrt{\omega_r^2 - \beta^2}$.

The Inertial tensor can be written as $\{I\} = \sum_\alpha m_\alpha\left[\delta_{ij}\vec{r}_\alpha^2 - \vec{r}_{\alpha,i}\vec{r}_{\alpha,j}\right]$

$\vec{F}_B = \vec{F}_{\text{inert}} - 2m\vec{\omega}\times\vec{v}_B - m\dot{\vec{\omega}}\times\vec{r}_B - m\vec{\omega}\times(\vec{\omega}\times\vec{r}_B)$, and $\vec{v}_{\text{inert}} = \vec{v}_B + \vec{\omega}\times\vec{r}_B$

$\vec{B} = \vec{\nabla}\times\vec{A}$; $\quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial ct}$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$; $\quad \cos(\alpha-\beta) = \sin\alpha\sin\beta + \cos\alpha\cos\beta$

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c}$

Integrals

For $c>0$ we have:

$$\int e^{cx}dx = \frac{1}{c}e^{cx} \ ; \quad \int x\, e^{cx}dx = \frac{cx-1}{c^2}e^{cx} \ ; \quad \int x^2 e^{cx}dx = \frac{c^2x^2-2cx+2}{c^3}e^{cx}$$